Ordering Properties of the Smallest Claim Amount from Two Heterogeneous Generalized Exponential Portfolios and their Application to Insurance

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Abstract

Suppose $X_{\lambda_1}, \ldots, X_{\lambda_n}$ is a set of non-negative random variables with X_{λ_i} having the distribution function generalized exponential, for $i = 1, \ldots, n$, and $I_{p_1}, \ldots, I_{p_n}$ are independent Bernoulli random variables, independent of the X_i's, with $E(I_{p_i}) = p_i$, $i = 1, \ldots, n$. Let $Y_i = X_{\lambda_i} I_{p_i}$, for $i = 1, \ldots, n$. It is of interest to note that in actuarial science, it corresponds to the claim amount in a portfolio of risks. In this paper, it's been tried to discuss the stochastic comparison between the smallest claim amounts in the sense of the usual stochastic order using the concept of vector weakly submajorization and under certain conditions. We obtain the usual stochastic order between the smallest claim amounts when the matrix of parameters (α, λ) changes to another matrix in a mathematical sense and finds an upper bound for the survival function of smallest claim amount. The results established here extend some well-known results in the literature and show that larger stochastic order smallest claim amount lead to the desirable property of uniformly larger Value-at-Risk.

Keywords: *Smallest Claim Amounts, Value-at-Risk, Generalized Exponential Distribution, Weakly Sunmajorization, Matrix Majorization, Schur-Convexity, Schur-Concavity.*

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1. Introduction

The generalized exponential distribution, as a generalization of the exponential distribution, has been discussed in reliability theory, life-testing and actuarial science. A random variable X is said to have the generalized exponential distribution with shape parameters $\alpha > 0$ and scale parameter $\lambda > 0$ (denoted by $X \sim GE(\alpha, \lambda)$), if its cumulative distribution and probability density functions are given by

$$F(x;\alpha,\lambda) = \left(1 - e^{-\lambda x}\right)^{u}, \qquad x > 0$$

and

$$f(x;\alpha,\lambda) = \alpha\lambda e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{\alpha - 1}, \qquad x > 0$$

respectively. The hazard rate function corresponding to X has the form

$$r(x;\alpha,\lambda) = \frac{\alpha\lambda e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{\alpha-1}}{1 - \left(1 - e^{-\lambda x}\right)^{\alpha}} \qquad x > 0$$

and admits different forms for different choices of the shape parameters. For more details on general properties of the generalized exponential distribution and its applications, interested readers may refer to Gupta and Kundu (2003, 2004), and Kundu et al. (2005).

Suppose X_{λ_i} denotes the total random claims that can be made in an insurance period, and I_{p_i} denotes a Bernoulli random variable associated with X_{λ_i} defined as follows:

$I_{p_i} = 1$ whenever the i[th] policyholder makes random claim X_{λ_i} and $I_{p_i} = 0$ whenever he/she does not make a claim. In actuarial science, $Y_i = X_{\lambda_i} I_{p_i}$ corresponds to the claim amount in a portfolio of risks.

Annual premium is the amount paid by the policyholder as the cost of the insurance cover being purchased. Indeed, it is the primary cost to the policyholder in transferring the risk to the insurer which depends on the type of insurance (life, health, auto, etc). In this regard, the smallest and largest claim amounts can have an important role in insurance analysis since they provide useful information to determine the annual premium.

It is quite important for the actuaries to be able to express preferences between random future gains or losses. For this purpose, stochastic ordering results become very useful. Stochastic orders have been used in various areas including management science, financial economics, insurance, actuarial science, operation research, reliability theory, queuing theory and survival analysis. For example, in the decision theory of financial economics, we can assist an individual in making proper decisions by comparing the risks leading to different uncertain payments. Interested readers may refer to Müller and Stoyan (2002) and Shaked and Shanthikumar (2007) for comprehensive discussions on univariate and multivariate stochastic orders.

The problem of comparing the number of claims and aggregate claim amounts with respect to some well-known stochastic orders is of interest from both theoretical and practical viewpoints. Several authors have worked on this problem; see, for example, Karlin and Novikoff (1963), Ma (2000), Frostig (2001), Hu and Ruan (2004), Denuit and Frostig (2006), Khaledi and Ahmadi (2008). Recently, Barmalzan et al. (2015) presented a complete version of the results of Khaledi and Ahmadi (2008) for aggregate claim amounts which have been extended to a more general case.

In this paper, the results established are developed in two directions. First, Suppose $X_{\lambda_1},\ldots,X_{\lambda_n}$ is a set of non-negative random variables with $X_{\lambda_i} \sim GE(\alpha,\lambda_i)$ for $i=1,\ldots,n$; and $I_{p_1},\ldots,I_{p_n}$ are independent Bernoulli random variables, independent of the X_{λ_i}'s, with $E\left(I_{p_i}\right)=p_i$, $i=1,\ldots,n$. Assume that $Y_{1:n}$, $Y_{1:n}^*$ denote the smallest claim amounts arising from $Y_i = X_{\lambda_1} I_{p_1}$, and $Y_i^* = X_{\lambda_i^*} I_{p_i^*}$, $i=1,\ldots,n$, and the following conditions hold:

(i) $\prod_{i=1}^{n} p_i \le \prod_{i=1}^{n} p_i^k$

(ii) $\left(\log\lambda_1,\ldots,\log\lambda_n\right) \succcurlyeq_w \left(\log\lambda_1^*,\ldots,\log\lambda_n^*\right)$

Then for $\alpha \ge 1$, we have $Y_{1:n}^* \ge_{st} Y_{1:n}$.

Next, Let us set

$$\mathcal{F}_n = \left\{ (\boldsymbol{\nu}, \boldsymbol{\tau}) = \begin{bmatrix} \nu_1 & \dots & \nu_n \\ \tau_1 & \dots & \tau_n \end{bmatrix} ;\ \nu_i \geq 1 ,\quad \tau_i \geq 0 \ and\ (\nu_i - \nu_j)(\tau_i - \tau_j) \leq 0, i, j = 1, \dots, n \right\}.$$

Suppose $X_{\lambda_1}, \dots, X_{\lambda_n}$ are independent non-negative random variables with $X_{\lambda_i} \sim GE(\alpha_i, \lambda_i)$, $i = 1, \dots, n$. Further, suppose $I_{p_1}, \dots, I_{p_n}$ are independent Bernoulli random variables, independent of the X_{λ_i}'s, with $E(I_{p_i}) = p_i$, $i = 1, \dots, n$. If $\prod_{i=1}^{n} p_i \leq \prod_{i=1}^{n} p_i^*$, then we obtain the following results from comparison of the smallest claim amounts:

(A_1) For $(\boldsymbol{\alpha}, \boldsymbol{\lambda}) \in \mathcal{F}_2$ and $(\boldsymbol{\alpha}^*, \boldsymbol{\lambda}^*) \in \mathcal{F}_2$, we have

$$\begin{bmatrix} \alpha_1 & \alpha_2 \\ \lambda_1 & \lambda_2 \end{bmatrix} \gg \begin{bmatrix} \alpha_1^* & \alpha_2^* \\ \lambda_1^* & \lambda_2^* \end{bmatrix} \Rightarrow Y_{1:2}^* \geq_{st} Y_{1:2}.$$

We also consider some generalizations of the case A_1 to the case when $n > 2$ as follows:

(A_2) If the T-transform matrices $T_1, \dots, T_k$ have the same structures, then for $(\boldsymbol{\alpha}, \boldsymbol{\lambda}) \in \mathcal{F}_n$, we have

$$\begin{bmatrix} \alpha_1^* & \dots & \alpha_n^* \\ \lambda_1^* & \dots & \lambda_n^* \end{bmatrix} = \begin{bmatrix} \alpha_1 & \dots & \alpha_n \\ \lambda_1 & \dots & \lambda_n \end{bmatrix} T_1 \dots T_k \Rightarrow Y_{1:n}^* \geq_{st} Y_{1:n}.$$

(A_3) Assume $(\boldsymbol{\alpha}, \boldsymbol{\lambda}) \in \mathcal{F}_n$ and $(\boldsymbol{\alpha}, \boldsymbol{\lambda}) T_1 \dots T_i \in \mathcal{F}_n$ for $i = 1, \dots, k - 1$, where $k \geq 2$. We then have

$$\begin{bmatrix} \alpha_1^* & \dots & \alpha_n^* \\ \lambda_1^* & \dots & \lambda_n^* \end{bmatrix} = \begin{bmatrix} \alpha_1 & \dots & \alpha_n \\ \lambda_1 & \dots & \lambda_n \end{bmatrix} T_1 \dots T_k \Rightarrow Y_{1:n}^* \geq_{st} Y_{1:n}.$$

Under the above setting, we can also show that larger stochastic order smallest claim amount leads to the desirable property of uniformly larger Value-at- Risk.

The rest of this paper is organized as follows:

In Section 2, we introduce some definitions and notations pertinent to stochastic orders, vector majorization, matrix Majorization and related orders.

Section 3 discusses stochastic comparisons of smallest claim amounts with independent heterogeneous in generalized exponential distribution.

In Section 4, we establish some ordering results relating to stochastic comparisons of the smallest claim amounts via matrix majorization. The results established here extend to some well-known results in the literature.

2. Preliminaries

In this section, we present the definitions of some well-known concepts relating to stochastic orders and majorization that are most pertinent to the results established in the subsequent sections. Based on the fact that a risk is smaller or less risky than another, one may deduce that it is also preferable in the mean-variance order that is used quite generally. In this ordering, one prefers the risk with the smaller mean, and the variance serves as a tie-breaker. This ordering concept, however, is not sufficient for actuarial purposes, since it leads to decisions about the attractiveness of risks about which there is no consensus among a group of sensible decision makers.

2.1. Stochastic Orders

Suppose X and Y are two non-negative continuous random variables with distribution functions $F(x) = P(X \le x)$ and $G(x) = P(Y \le x)$, and survival functions $\bar{F}(x) = 1 - F(x)$ and $\bar{G}(x) = 1 - G(x)$, and stop-loss transforms $\pi_X(d) = E(X-d)^+ = E(max(0, X - d))$ and $\pi_Y(d) = E(Y-d)^+ = E(max(0, Y-d))$, $d > 0$, respectively.

Definition 1 (i) X is said to be larger than Y in the usual stochastic order (denoted by $\ge_{st} Y$) if $\bar{F}(x) \ge \bar{G}(x)$ for all $x > 0$. This is equivalent to saying that $E\big(\varphi(X)\big) \ge E(\varphi(Y))$ for all increasing functions $\varphi: \mathcal{R}^+ \to \mathcal{R}$ when the involved expectations exist;

(ii) X is said to be larger than Y in the stop-loss order, or equivalently the increasing convex order (denoted by $X \ge_{sl} Y$) if $\pi_X(d) \ge \pi_Y(d)$ for all $d > 0$, when the involved expectations exist.

Note that the usual stochastic order implies the stop-loss order. Stop-loss order represents the common preferences of all risk averse decision makers. Stop-loss order applies to losses, that is, non-negative risks. The stop-loss transform and order have been discussed extensively in the actuarial literature, and have been used widely in casualty and health insurance to compare underlying risks. Goovaerts et al. (1984, 1990), Kaas et al. (1994), Kaas and Hesselager (1995) and Hürlimann (2000) all have studied the stop-loss transform and order.

2.2. Majorization Order

The degree of heterogeneity of the portfolio, reflected in the vector of parameters, can then be compared by using an appropriate order relation for vectors of real numbers, like the majorization order. Majorization is a way of comparing two non-negative vectors (of the same dimension) in terms of the dispersion of their components. When majorization inequality holds between two vectors, the portfolio with the largest vector often turns out to be the more heterogeneous one.

Definition 2 For two vectors $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ and $\boldsymbol{\lambda}^* = (\lambda_1^*, \dots, \lambda_n^*)$, suppose $\lambda_{(1)} < \cdots < \lambda_{(n)}$ and $\lambda_{(1)}^* < \cdots < \lambda_{(n)}^*$ denote the increasing arrangements of their components, respectively. Then

(i) a vector $\boldsymbol{\lambda}$ is said to be majorized by another vector $\boldsymbol{\lambda}^*$ (denoted by $\boldsymbol{\lambda} \succcurlyeq^{\boldsymbol{m}} \boldsymbol{\lambda}^*$) if $\sum_{j=1}^{i} \lambda_{(j)} \le \sum_{j=1}^{i} \lambda_{(j)}^*$, for $i = 1, \dots, n-1$, and $\sum_{j=1}^{n} \lambda_{(j)} \le \sum_{j=1}^{n} \lambda_{(j)}^*$;

(ii) a vector $\boldsymbol{\lambda}$ is said to be weakly submajorized by another vector $\boldsymbol{\lambda}^*$ (denoted by $\boldsymbol{\lambda} \ge_{\boldsymbol{w}} \boldsymbol{\lambda}^*$) if $\sum_{j=1}^{i} \lambda_{(j)} \ge \sum_{j=1}^{i} \lambda_{(j)}^*$, for $i = 1, \dots, n$.

The concept of majorization is a way of comparing two vectors of the same dimension, in terms of the dispersion of their components for which the order $\boldsymbol{\lambda} \succcurlyeq^{\boldsymbol{m}} \boldsymbol{\lambda}^*$ results in the λ_i's being more diverse than the λ_i^* 's, for a fixed sum. For example, we always have $\boldsymbol{\lambda} \succcurlyeq^{\boldsymbol{m}} \bar{\boldsymbol{\lambda}}$, where $\bar{\boldsymbol{\lambda}} = (\bar{\lambda}, \dots, \bar{\lambda})$ with $\bar{\lambda} = n^{-1} \sum_{i=1}^{n} \lambda_i$. It is evident that the majorization order implies weak submajorization order.

The next lemma provides some conditions for the characterization of Schur-convex and Schur-concave functions.

Lemma 1 (Marshall et al. (2011), p. 84) Consider the real-valued continuously differentiable function φ on I^n, where $I^n \subset \mathcal{R}$ is an open interval. Then, φ is Schur-convex on I^n if and only if

(i) φ is symmetric on I^n, and

(ii) for all $i \neq j$ and all $z \in I^n$

$$(z_i - z_j)\left(\frac{\partial\varphi}{\partial z_i}(\boldsymbol{z}) - \frac{\partial\varphi}{\partial z_j}(\boldsymbol{z})\right) \geq 0,$$

where $\frac{\partial\varphi}{\partial z_i}(\boldsymbol{z})$ denotes the partial derivative of φ with respect to its i-th argument. If the side of inequality is reversed then φ is said to be Schur-concave.

The following result provides some conditions for the characterization of vector functions that preserve weak submajorization orders.

Lemma 2 (Marshall et al. (2011), p. 87) Consider the real-valued function φ defined on a set $\subset \mathcal{R}^n$. Then, $\boldsymbol{\lambda} \geq_w \boldsymbol{\lambda}^*$ implies $\varphi(\boldsymbol{\lambda}) \geq \varphi(\boldsymbol{\lambda}^*)$ if and only if φ is increasing and Schur-convex on A.

Several generalizations of the concept of majorization have been introduced in the literature, and one of them is majorization for matrices. This is motivated by the theorem of Hardy-Littlewood-P´olya (see Chapter 2 of Marshall et al. (2011)). A square matrix π_n, of order n is said to be a permutation matrix if each row and column has a single entry as 1, and all other entries as zero. Such matrices are obtained by interchanging rows (or columns) of the identity matrix In. The T-transform matrix has the form $T = \delta I_n + (1-\delta)\,\pi_\mathrm{n}$, where $0 \leq \delta \leq 1$ and π_n is a permutation matrix that just interchanges two coordinates. It should be mentioned here that permutation matrices used in the T-transform matrices are the identity matrix I_n with its two columns interchanged. We say that two T-transform matrices $T_1 = \delta_1 I_n + (1-\delta_1)\pi_\mathrm{n}$ and $T_1 = \delta_2 I_n + (1-\delta_2)\pi_\mathrm{n}$ have the same structure if $\pi_1 = \pi_2$. It is well-known that the finite product of T-transform matrices with the same structures is also a T-transform matrix, while this product may not be a T-transform matrix if its elements do not have the same structure; see Balakrishnan et al. (2014) for more details. The following definition presents two generalizations of the vector majorization.

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Definition 3 Suppose $U = \{u_{ij}\}$ and $V = \{v_{ij}\}$ are $m \times n$ matrices, with $u_1^R, \dots, u_m^R$, and $v_1^R, \dots, v_m^R$ as the rows of U and V, respectively. Then,

(i) V is said to be row majorized by U (denoted by $U >^{row} V$) if $u_i^R \succ^m v_i^R$, for $i = 1, \dots, m$;

(ii) V is said to be chain majorized by U (denoted by $U \gg V$) if there exists a finite set of $n \times n$ T_i-transform matrices, $i = 1, \dots, k$, such that $V = UT_1 \dots T_k$.

The following implications between these multivariate maorizations are well-known:

$$U \gg V \Rightarrow U >^{row} V.$$

Main Results

3. Stochastic Order via Submajorization

This section discusses some conditions for the comparison of the smallest claim amounts, with respect to weakly submajorization order. Suppose $X_{\lambda_1}, \dots, X_{\lambda_n}$ are independent non-negative random variables with $X_{\lambda_i} \sim GE(\alpha, \lambda_i)$, $i = 1, \dots, n$, where $\lambda_i > 0$. Further, suppose $I_{p_1}, \dots, I_{p_n}$ are independent Bernoulli random variables, independent of the X_{λ_i}'s, with $E(I_{p_i}) = p_i$, $i = 1, \dots, n$. Set $Y_i = X_{\lambda_i} I_{p_i}$, for $i = 1, \dots, n$. Then, the survival function of $Y_{1:n} = min(Y_1, \dots, Y_n)$, the smallest claim amount, is given by

$$\bar{F}_{Y_{1:n}}(x) = P(Y_{1:n} > x)$$

$$= P\left(X_{\lambda_1} I_{p_1} > x, \dots, X_{\lambda_n} I_{p_n} > x\right)$$

$$= P\left(X_{\lambda_1} I_{p_1} > x, \dots, X_{\lambda_n} I_{p_n} > x | I_{p_1} = 1, \dots, I_{p_n} = 1\right) P\left(I_{p_1} = 1, \dots, I_{p_n} = 1\right)$$

$$= P\left(X_{\lambda_1} > x, \dots, X_{\lambda_n} > x\right) P\left(I_{p_1} = 1, \dots, I_{p_n} = 1\right)$$

$$= \prod_{i=1}^{n} P\left(X_{\lambda_i} > x\right) \; P\left(I_{p_i} = 1\right)$$

$(\prod_{i=1}^{n} p_i)\{\prod_{i=1}^{n}\left(1-\left(1-e^{-\lambda_i x}\right)^{\alpha}\right)\}, \qquad x \geq 0. \qquad (1)$

We now establish two lemmas that become quite useful for obtaining the usual stochastic order between smallest claim amounts.

Lemma 3: Let $X \sim GE(\alpha,,1)$ with hazard rate r. Then $xr(x)$ is increasing in $x \in \mathcal{R}^+$, for $\alpha \geq 1$.

Proof: It can be easily seen that $xr(x) = \Phi(1-e^{-x})$, where

$$\Phi(t) = (-\,ln(1-t))\ \frac{\alpha(1-t)t^{\alpha-1}}{1-t^{\alpha}}$$

$$0 < t < 1\,, \alpha \geq 1,$$

It is easy to show that $(1-t)t^{\alpha-1}/1-t^{\alpha}$ is increasing in $t \in (0,1)$, for $\alpha \geq 1$. Furthermore, the non-negative function $(-\,ln(1-t))$ is also increasing in $t \in (0,1)$. Since the product of two non-negative increasing functions is also increasing, we find that $\Phi(t)$ is increasing in $t \in (0,1)$ for $\alpha \geq 1$, which completes the proof of the lemma.

The following lemma provides a usual stochastic order between smallest order statistics from two heterogeneous generalized exponential distributions.

Lemma 4: Suppose $X_{\lambda_1}, \dots, X_{\lambda_n}$ are independent non-negative random variables with $X_{\lambda_i} \sim GE(\alpha, \lambda_i)$, where $\lambda_i > 0, i = 1, \dots, n$. Further, suppose $I_{p_1}, \dots, I_{p_n}$ are independent Bernoulli random variables, independent of the X_{λ_i}'s, with $E\left(I_{p_i}\right) = p_i$, $i = 1, \dots, n$. Then for $\alpha \geq 1$,

$$(\log\lambda_1, \dots, \log\lambda_n) \succcurlyeq_w (\log\lambda_1^*, \dots, \log\lambda_n^*) \Rightarrow X_{1:n}^* \geq_{st} (\geq_{sl}) X_{1:n}.$$

Proof: The survival function of $X_{1:n}$ can be written as

$$\bar{F}_{X_{1:n}}(x) = \prod_{i=1}^{n}\left(1-\left(1-e^{-x\,e^{b_i}}\right)^{\alpha}\right), \qquad x > 0,$$

where $b_i = \log\lambda_i$, for $i = 1, \dots, n$. By Lemma 2, the desired result follows if $\bar{F}_{X_{1:n}}(x)$ is decreasing and Schur-concave in $(b, \dots, b_n)$, for all fixed $x > 0$. Taking derivative of $\bar{F}_{X_{1:n}}(x)$ with respect to b_i, we obtain

$$\frac{\partial \bar{F}_{X_{1:n}}(x)}{\partial b_i} = -x\, e^{b_i} \bar{F}_{X_{1:n}}(x)\, r(x\, e^{b_i}).$$

Clearly, the above derivative is negative which results in $\bar{F}_{X_{1:n}}(x)$ being decreasing in $(b_1, \dots, b_n)$. On the other hand, we have

$$\left(b_i - b_j\right)\left(\frac{\partial \bar{F}_{X_{1:n}}(x)}{\partial b_i} - \frac{\partial \bar{F}_{X_{1:n}}(x)}{\partial b_j}\right)$$

$$= x\bar{F}_{X_{1:n}}(x)\left(b_i - b_j\right)\left(e^{b_j}\, r\left(x\, e^{b_j}\right) - e^{b_i}\, r(x\, e^{b_i})\right) \qquad (2)$$

From Lemma 3, because $xr(x)$ is increasing in x, it readily follows that $e^b\, r(xe^b)$ is also increasing in a, and so the right hand side of (2) is non-positive. This result, along with Lemma 1, implies that $\bar{F}_{X_{1:n}}(x)$ is Schur-concave in $(b_1, \dots, b_n)$, thus completing the proof of the Lemma.

In the following theorem, we discuss stochastic comparison between the smallest claim amounts in the sense of the usual stochastic order.

Theorem 2: Suppose $X_{\lambda_1}, \dots, X_{\lambda_n}$ are independent non-negative random variables with $X_{\lambda_i} \sim GE(\alpha, \lambda_i)$, where $\lambda_i > 0$, $i = 1, \dots, n$. Further, suppose $I_{p_1}, \dots, I_{p_n}$ are independent Bernoulli random variables, independent of the X_{λ_i}'s, with $E\left(I_{p_i}\right) = p_i$, $i = 1, \dots, n$. Assume that the followingconditions hold:

(i) $\prod_{i=1}^{n} p_i \leq \prod_{i=1}^{n} p_i^*$;

(ii) $(\log \lambda_1, \dots, \log \lambda_n) \succcurlyeq_w (\log \lambda_1^*, \dots, \log \lambda_n^*)$;

Then for $\alpha \geq 1$, we have $Y_{1:n}^* \geq_{st} (\geq_{sl}) Y_{1:n}$.

Proof. By Lemma 4, assumption (ii) implies $X_{1:n}^* \geq_{st} X_{1:n}$ for $\alpha \geq 1$. Now, in light of this observation and (1), the required result follows immediately from assumption (i). ■

The Value-at-Risk, denoted by VaR, which is defined based on quantiles of a random variable plays a critical role in risk measurement; see Jorion (2000) for more details. Two random risks X and Y can be compared by means of their VaR's. We may have two probability levels p_0 and p_1 such that $VaR[X; p_0] \geq VaR[Y;\, p_0]$ and

$VaR[Y; p_1] \geq VaR[X; p_1]$. So, it is reasonable to consider a situation under which $VaR[X; p] \geq VaR[Y; p]$ for all probability level $p \in (0, 1)$. For a risk X, the Value-at-Risk (VaR) at level p is defined as

$$VaR[X; p] = F^{-1}(p) = inf\ \{u: F(u) \geq p\}.$$

It is of interest to note that the usual stochastic order implies Value-at-Risk. The following corollary is a direct consequence of Theorem 2 which shows that larger stochastic order smallest claim amount leads to the desirable property of uniformly larger VaR.

Corollary 1: Suppose $X_{\lambda_1}, \ldots, X_{\lambda_n}$ are independent non-negative random variables with $X_{\lambda_i} \sim GE(\alpha, \lambda_i)$, $i = 1, \ldots, n$, where $\lambda_i > 0$, $i = 1, \ldots, n$. Then, suppose $I_{p_1}, \ldots, I_{p_n}$ are independent Bernoulli random variables, independent of the X_{λ_i}'s, with $E(I_{p_i}) = p_i$, $i = 1, \ldots, n$. Assume that the followingconditions hold:

(i) $\prod_{i=1}^{n} p_i \leq \prod_{i=1}^{n} p_i^*$;

(ii) $(\log \lambda_1, \ldots, \log \lambda_n) \succcurlyeq_w (\log \lambda_1^*, \ldots, \log \lambda_n^*)$;

Then for $\alpha \geq 1$, we have $VaR[Y_{1:n}^*; p] \geq VaR[Y_{1:n}; p]$. ■

4. Stochastic Order via Matrix Majorization

In this section, we show that the usual stochastic order between smallest claim amounts when the matrix of parameters $(\boldsymbol{\alpha}, \boldsymbol{\lambda})$ changes to another matrix in a mathematical sense. For this purpose, we need the following lemmas.

Lemma 5: (Balakrishnan et al. (2013)) Suppose the function $u: (0,1) \times (0,1) \to (-\infty, 0)$ is defined as

$$u(\alpha, z) = \frac{z^{\alpha} lnz}{1 - z^{\alpha}}.$$

Then,

(i) for each $0 < z < 1$, the function $u(\alpha, z)$ is increasing with respect to α.

(ii) for each $\alpha > 0$, the function $u(\alpha, z)$ is decreasing with respect to z. ■

Lemma 6: (Balakrishnan et al. (2013)) Suppose the function $k: (0, \infty) \times (0, 1) \to (0, \infty)$ is defined as

$$k(\alpha, z) = \frac{\alpha\,(1 - z)\, z^{\alpha - 1}}{1 - z^{\alpha}} \ .$$

Then,

(i) for each $0 < z < 1$, the function $k(\alpha, z)$ is decreasing with respect to α.

(ii) for each $0 < \alpha \le 1$, the function $u(\alpha, z)$ is decreasing with respect to z, and for each $\alpha \ge 1$, the function $u(\alpha, z)$ is increasing with respect to z. ■

Let us set

$$\mathcal{F}_n = \left\{ (\boldsymbol{\nu}, \boldsymbol{\tau}) = \begin{bmatrix} \nu_1 & \cdots & \nu_n \\ \tau_1 & \cdots & \tau_n \end{bmatrix} ;\ \nu_i \ge 1\,,\quad \tau_i \ge 0 \ and\ \left(\nu_i - \nu_j\right)\left(\tau_i - \tau_j\right) \le 0, i, j = 1, \dots, n \right\}.$$

The proof of the two next lemmas are quite similar to those of Theorems 2 and 3 of Balakrishnan et al. (2014), and are therefore omitted here for the sake of brevity.

Lemma 7: Consider the differentiable function $\varphi: \mathcal{R}^{+4} \to \mathcal{R}^{+}$: Then,

$$\varphi(U) \ge \varphi(V)$$

$$for\ all\ matrices\ \ U, V\ such\ \ that\ U \in \mathcal{F}_2\ and\ \ U \gg V, \tag{3}$$

if and only if

(i) $\varphi(U) = \varphi(U\Pi)$ for all permutation matrices Π and $U \in \mathcal{F}_2$;

(ii) $\sum_{i=1}^{m} \left(u_{ik} - u_{ij}\right) \left(\varphi_{ik}(U) - \varphi_{ij}(U)\right) \ge (\le) 0$, for all $j, k = 1, 2$ where $\varphi_{ij}(U) = \frac{\partial \varphi(U)}{\partial u_{ij}}$. ■

Lemma 8: Consider the differentiable function $\varphi: \mathcal{R}^{+2} \to \mathcal{R}^{+}$; and let the function $\varphi_n: \mathcal{R}^{+2} \to \mathcal{R}^{+}$ be defined as $\varphi_n(U) = \prod_{i=1}^{n} \varphi(u_{1i}, u_{2i})$ If φ_2 satisfies (3), then $\varphi_n(U) \ge \varphi_n(V)$ where $U \in \mathcal{F}_n$ and $V = UT$. ■

In the following lemmas, we discuss stochastic comparison between the smallest claim amounts in the sense of the usual stochastic order via matrix majorization.

Lemma 9: Suppose $X_{\lambda_1}, X_{\lambda_2}$ are independent non-negative random variables with $X_{\lambda_i} \sim GE(\alpha_i, \lambda_i)$, $i = 1, 2$. Then for $(\boldsymbol{\alpha}, \boldsymbol{\lambda}) \in \mathcal{F}_2$ and $(\boldsymbol{\alpha}^*, \boldsymbol{\lambda}^*) \in \mathcal{F}_2$, we have

$$\begin{bmatrix} \alpha_1 & \alpha_2 \\ \lambda_1 & \lambda_2 \end{bmatrix} \gg \begin{bmatrix} \alpha_1^* & \alpha_2^* \\ \lambda_1^* & \lambda_2^* \end{bmatrix} \Rightarrow X_{1:2}^* \geq_{st} (\geq_{sl}) X_{1:2}.$$

Proof. The survival function of $X_{1:2}$ is

$$\bar{F}_{X_{1:2}}(x) = \prod_{i=1}^{2} \left(1 - \left(1 - e^{-\lambda_i x}\right)^{\alpha_i}\right) \qquad x > 0,$$

It is clear that for fixed $x > 0$, the function $\bar{F}_{X_{1:2}}(x)$ is symmetric in (α_i, λ_i)'s and so Condition (i) of Theorem 1 is satisfied. For fixed $x > 0$ and $i \neq j$, consider the function φ as

$$\varphi(\boldsymbol{\alpha}, \boldsymbol{\lambda}) = (\alpha_1 - \alpha_2)\left(\frac{\partial \bar{F}_{X_{1:2}}(x)}{\partial \alpha_1} - \frac{\partial \bar{F}_{X_{1:n}}(x)}{\partial \alpha_2}\right)$$

$$+(\lambda_1 - \lambda_2)\left(\frac{\partial \bar{F}_{X_{1:2}}(x)}{\partial \lambda_1} - \frac{\partial \bar{F}_{X_{1:2}}(x)}{\partial \lambda_2}\right) \tag{4}$$

The partial derivative of $\bar{F}_{X_{1:2}}(x)$ with respect to α_i is

$$\frac{\partial \bar{F}_{X_{1:2}}(x)}{\partial \alpha_i} = -\bar{F}_{X_{1:2}}(x) \frac{ln\left(1 - e^{-\lambda_i x}\right)\left(1 - e^{-\lambda_i x}\right)^{\alpha_i}}{1 - (1 - e^{-\lambda_i x})^{\alpha_i}}$$

$$= \; -\bar{F}_{X_{1:2}}(x)\, u\left(\alpha_i, 1 - e^{-\lambda_i x}\right), \tag{5}$$

where $u(\alpha, t)$ is as defined in Lemma 5. Further, the partial derivative of $\bar{F}_{X_{1:2}}(x)$ with respect to λ_i is

$$\frac{\partial \bar{F}_{X_{1:2}}(x)}{\partial \lambda_i} = -x\, \bar{F}_{X_{1:2}}(x)\, k\left(\alpha_i, 1 - e^{-\lambda_i x}\right), \tag{6}$$

where $k(\alpha, t)$ is as defined in Lemma 6. Now, upon substituting (5) and (6) in (4), we get

$$\varphi(\boldsymbol{\alpha}, \boldsymbol{\lambda}) = \bar{F}_{X_{1:2}}(x)(\alpha_1 - \alpha_2)\left(u\left(\alpha_2, 1 - e^{-\lambda_2 x}\right) - u\left(\alpha_1, 1 - e^{-\lambda_1 x}\right)\right) +$$

$$x\, \bar{F}_{X_{1:2}}(x)(\lambda_1 - \lambda_2)\left(k\left(\alpha_2, 1 - e^{-\lambda_2 x}\right) - k\left(\alpha_1, 1 - e^{-\lambda_1 x}\right)\right), \tag{7}$$

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According to the assumption that $(\boldsymbol{\alpha},\boldsymbol{\lambda}) \in \mathcal{F}_2$, we have $\alpha_1 \geq \alpha_2 \geq 1$ and $\lambda_2 \geq \lambda_1$ or $\alpha_2 \geq \alpha_1 \geq 1$ and $\lambda_1 \geq \lambda_2$. Here, we present the proof only for case $\alpha_1 \geq \alpha_2 \geq 1$ and $\lambda_2 \geq \lambda_1$, since the proof for the other case is quite similar. Based on Lemma 5, we observe that $u(\alpha, t)$ is increasing with respect to α for fixed t, and is decreasing with respect to t for fixed α. So, from these observations, we conclude that

$$u\left(\alpha_1, 1-e^{-\lambda_1 x}\right) \geq u\left(\alpha_2, 1-e^{-\lambda_1 x}\right) \geq u\left(\alpha_2, 1-e^{-\lambda_2 x}\right),$$

which together with the assumption that $\alpha_1 \geq \alpha_2$, implies that the first term on the right hand side of (7) is non-positive. On the other hand, from Lemma 6, it follows that $k(\alpha, t)$ is decreasing with respect to α for fixed t, and is increasing with respect to t for fixed α. Based on the assumptions $\alpha_1 \geq \alpha_2 \geq 1$ and $\lambda_2 \geq \lambda_1$, once again, we have

$$k\left(\alpha_2, 1-e^{-\lambda_2 x}\right) \geq k\left(\alpha_1, 1-e^{-\lambda_2 x}\right) \geq k\left(\alpha_1, 1-e^{-\lambda_1 x}\right), \tag{8}$$

by using (8), we observe that the second term on the right hand side of (7) is also non-positive. Therefore, Condition (ii) of Theorem 1 is satisfied, and the proof is thus completed. ■

Some generalizations of the result in Lemma 9 to the case when the number of underlying random variables is arbitrary are presented below.

Lemma 10: Suppose $X_{\lambda_1}, \dots, X_{\lambda_n}$ are independent non-negative random variables with $X_{\lambda_i} \sim GE(\alpha_i, \lambda_i)$, $i = 1, \dots, n$. Then, for $(\boldsymbol{\alpha},\boldsymbol{\lambda}) \in \mathcal{F}_n$, we have

$$\begin{bmatrix} \alpha_1^* & \dots & \alpha_n^* \\ \lambda_1^* & \dots & \lambda_n^* \end{bmatrix} = \begin{bmatrix} \alpha_1 & \dots & \alpha_n \\ \lambda_1 & \dots & \lambda_n \end{bmatrix} T \Rightarrow X_{1:n}^* \geq_{st} (\geq_{sl}) \, X_{1:n}.$$

Proof: Setting $\varphi_n(\boldsymbol{\alpha},\boldsymbol{\lambda}) = \bar{F}_{X_{1:n}}(x)$ and $\varphi(\alpha,\lambda) = \left(1-\left(1-e^{-\lambda_i x}\right)^{\alpha_i}\right)$, for fixed $x > 0$, we then have $\varphi_n(\boldsymbol{\alpha},\boldsymbol{\lambda}) = \prod_{i=1}^{n}\left(1-\left(1-e^{-\lambda_i x}\right)^{\alpha_i}\right)$. According to Lemma 9, φ_2 is satisfied in (3). Now, the required result follows immediately from Lemma

7. ■

Lemma 11: Suppose $X_{\lambda_1},\dots,X_{\lambda_n}$ are independent non-negative random variables with $X_{\lambda_i} \sim GE(\alpha_i,\lambda_i)$, $i=1,\dots,n$. If the T-transform matrices $T_1,\dots,T_k$ have the same structures, then for $(\boldsymbol{\alpha},\boldsymbol{\lambda}) \in \mathcal{F}_n$, we have

$$\begin{bmatrix} \alpha_1^* & \dots & \alpha_n^* \\ \lambda_1^* & \dots & \lambda_n^* \end{bmatrix} = \begin{bmatrix} \alpha_1 & \dots & \alpha_n \\ \lambda_1 & \dots & \lambda_n \end{bmatrix} T_1 \dots T_k \Rightarrow X_{1:n}^* \geq_{st} (\geq_{sl}) X_{1:n}.$$

Proof. The desired result is immediately obtained from Lemma 10 and the fact that the finite product of T-transform matrices with the same structure is also a T-transform matrix. ■

It will be of interest to know whether the results of Lemma 11 will still be held if the matrices T_i, $i=1,\dots,k$, do not have the same structure. The following lemma gives an answer to this question.

Lemma 12 : Suppose $X_{\lambda_1},\dots,X_{\lambda_n}$ are independent non-negative random variables with $\mathrm{X}_{\lambda_\mathrm{i}} \sim \mathrm{GE}(\alpha_\mathrm{i},\lambda_\mathrm{i})$, $i=1,\dots,n$. Assume $(\boldsymbol{\alpha},\boldsymbol{\lambda}) \in \mathcal{F}_n$ and $(\boldsymbol{\alpha},\boldsymbol{\lambda})T_1 \dots T_i \in \mathcal{F}_n$ for $i=1,\dots,k-1$, where $k \geq 2$. Then, we have

$$\begin{bmatrix} \alpha_1^* & \dots & \alpha_n^* \\ \lambda_1^* & \dots & \lambda_n^* \end{bmatrix} = \begin{bmatrix} \alpha_1 & \dots & \alpha_n \\ \lambda_1 & \dots & \lambda_n \end{bmatrix} T_1 \dots T_k \Rightarrow X_{1:n}^* \geq_{st} (\geq_{sl}) X_{1:n}.$$

Proof. The required result can be easily obtained by repeating the result of Lemma 10 for the matrices $(\boldsymbol{\alpha},\boldsymbol{\lambda})T_1 \dots T_i \in \mathcal{F}_n$, for $i=1,\dots,k-1$. ■

By using Lemma 9-12, we obtain the following theorems which enable us to compare the smallest claim amounts in the sense of the usual stochastic order.

Theorem 3: Suppose X_{λ_1}, X_{λ_2} are independent non-negative random variables with $\mathrm{X}_{\lambda_\mathrm{i}} \sim \mathrm{GE}(\alpha_\mathrm{i},\lambda_\mathrm{i})$, $i=1,2$. Further, suppose I_{p_1}, I_{p_2} are independent Bernoulli random variables, independent of the X_{λ_i}'s, with $E(I_{p_i}) = p_i$, $i=1,2$. Assume that $(\boldsymbol{\alpha},\boldsymbol{\lambda}) \in \mathcal{F}_n$, and the following conditions hold:

(i) $\prod_{i=1}^n p_i \leq \prod_{i=1}^n p_i^*$;

(ii) $\begin{bmatrix} \alpha_1 & \alpha_2 \\ \lambda_1 & \lambda_2 \end{bmatrix} \gg \begin{bmatrix} \alpha_1^* & \alpha_2^* \\ \lambda_1^* & \lambda_2^* \end{bmatrix}$;

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Then, we have $Y_{1:2}^* \geq_{st} (\geq_{sl}) Y_{1:2}$.

Proof: By Lemma 9, assumption (ii) implies $X_{1:n}^* \geq_{st} X_{1:n}$. Now, in light of this observation and (1), the required result follows immediately from assumption (i). ■

Remark 1: Theorem 3 can be used to compute a lower bound for the survival function of the smallest claim amount based on a heterogeneous portfolio of risks in terms of the survival function of smallest claim amounts based on a homogeneous portfolio of risks. More precisely, considering the T-transform matrix $T = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$ and setting $\bar{\alpha} = \frac{\alpha_1+\alpha_2}{2}$, $\bar{\lambda} = \frac{\lambda_1+\lambda_2}{2}$, $\bar{p} = \frac{p_1+p_2}{2}$. We can easily observe

$$\begin{bmatrix} \bar{\alpha} & \bar{\alpha} \\ \bar{\lambda} & \bar{\lambda} \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \lambda_1 & \lambda_2 \end{bmatrix} \text{T},$$

which, according to Part (ii) of Definition 3, implies

$$\begin{bmatrix} \alpha_1 & \alpha_2 \\ \lambda_1 & \lambda_2 \end{bmatrix} \gg \begin{bmatrix} \bar{\alpha} & \bar{\alpha} \\ \bar{\lambda} & \bar{\lambda} \end{bmatrix},$$

Thus, by using these observations and Theorem 3 and this fact $p_1 p_2 \leq (\bar{p})^2$, we can obtain the following upper bound for the survival function of $Y_{1:2}$ based on that of $Y_{1:2}^*$:

$$\bar{F}_{X_{1:2}}(x) \leq (\bar{p})^2 \left(1 - \left(1 - e^{-\bar{\lambda}\, x}\right)^{\bar{\alpha}}\right)^2 . \blacksquare$$

Theorem 4: Suppose $X_{\lambda_1}, \ldots, X_{\lambda_n}$ are independent non-negative random variables with $X_{\lambda_i} \sim GE(\alpha_i, \lambda_i)$, $i = 1, \ldots, n$. Further, suppose $I_{p_1}, \ldots, I_{p_n}$ are independent Bernoulli random variables, independent of the X_{λ_i}'s, with $E\left(I_{p_i}\right) = p_i$, $i = 1, \ldots, n$. Assume that $(\boldsymbol{\alpha}, \boldsymbol{\lambda}) \in \mathcal{F}_n$, and the following conditions hold:

(i) $\prod_{i=1}^{n} p_i \leq \prod_{i=1}^{n} p_i^*$;

(ii) $\begin{bmatrix} \alpha_1^* & \ldots & \alpha_n^* \\ \lambda_1^* & \ldots & \lambda_n^* \end{bmatrix} = \begin{bmatrix} \alpha_1 & \ldots & \alpha_n \\ \lambda_1 & \ldots & \lambda_n \end{bmatrix} T \Rightarrow Y_{1:n}^* \geq_{st} Y_{1:n}$;

Then, we have $Y_{1:n}^* \geq_{st} (\geq_{sl}) Y_{1:n}$.

Proof: By Lemma 10, assumption (ii) implies $X_{1:n}^* \geq_{st} X_{1:n}$. Now, in light of this observation and (1), the required result follows immediately from assumption (i). ■

Theorem 5: Suppose $X_{\lambda_1}, \dots, X_{\lambda_n}$ are independent non-negative random variables with $X_{\lambda_i} \sim GE(\alpha_i, \lambda_i)$, $i = 1, \dots, n$. Further, suppose $I_{p_1}, \dots, I_{p_n}$ are independent Bernoulli random variables, independent of the X_{λ_i}'s, with $E(I_{p_i}) = p_i$, $i = 1, \dots, n$. If the T-transform matrices $T_1, \dots, T_k$ have the same structures, $(\boldsymbol{\alpha}, \boldsymbol{\lambda}) \in \mathcal{F}_n$ and the following conditions hold:

(i) $\prod_{i=1}^{n} p_i \leq \prod_{i=1}^{n} p_i^*$;

(ii) $\begin{bmatrix} \alpha_1^* & \dots & \alpha_n^* \\ \lambda_1^* & \dots & \lambda_n^* \end{bmatrix} = \begin{bmatrix} \alpha_1 & \dots & \alpha_n \\ \lambda_1 & \dots & \lambda_n \end{bmatrix} T_1 \dots T_k$;

Then, for $(\boldsymbol{\alpha}, \boldsymbol{\lambda}) \in \mathcal{F}_n$, we have $Y_{1:n}^* \geq_{st} (\geq_{sl}) Y_{1:n}$.

Proof: By Lemma 11, assumption (ii) implies $X_{1:n}^* \geq_{st} X_{1:n}$. Now, in light of this observation and (1), the required result follows immediately from assumption (i). ■

The following example provides an illustration of the result in Theorem 5.

Example 1: Suppose $X_{\lambda_1}, X_{\lambda_2}, X_{\lambda_3}$ are independent non-negative random variables with $X_{\lambda_i} \sim GE(\alpha_i, \lambda_i)$, $i = 1, 2, 3$. Further, suppose $I_{p_1}, I_{p_2}, I_{p_3}$ are independent Bernoulli random variables, independent of the X_{λ_i}'s, with $E(I_{p_i}) = p_i$, $i = 1, 2, 3$. Also, suppose

$(\alpha_1, \alpha_2, \alpha_3) = (2, 6, 4)$, $(\lambda_1, \lambda_2, \lambda_3) = (8, 4, 6)$, $(\alpha_1^*, \alpha_2^*, \alpha_3^*) = (2, 5.12, 4.88)$, $(\lambda_1^*, \lambda_2^*, \lambda_3^*) = (8, 4.88, 5.12)$, $(p_1, p_2, p_3) = (0.2, 0.3, 0.1)$, $(p_1^*, p_2^*, p_3^*) = (0.1, 0.4, 0.2)$.

Consider the T −transform matrices T_1 and T_2 as follows:

$$T_1 = 0.2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 0.8 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$T_2 = 0.4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 0.6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

It is then easy to observe that $(\boldsymbol{\alpha}, \boldsymbol{\lambda})$ in $\mathcal{F}_3$ and $(\boldsymbol{\alpha}^*, \boldsymbol{\lambda}^*) = (\boldsymbol{\alpha}, \boldsymbol{\lambda}) T_1 T_2$ which, based on Theorem 5, yields $Y_{1:3}^* \geq_{st} (\geq_{sl}) Y_{1:3}$. ■

Theorem 6: Suppose $X_{\lambda_1}, \dots, X_{\lambda_n}$ are independent non-negative random variables with $X_{\lambda_i} \sim GE(\alpha_i, \lambda_i)$, $i = 1, \dots, n$. Further, suppose $I_{p_1}, \dots, I_{p_n}$ are independent Bernoulli random variables, independent of the X_{λ_i}'s, with $E(I_{p_i}) = p_i$, $i = 1, \dots, n$. Assume $(\boldsymbol{\alpha}, \boldsymbol{\lambda}) \in \mathcal{F}_n$ and $(\boldsymbol{\alpha}, \boldsymbol{\lambda})T_1 \dots T_i \in \mathcal{F}_n$ for $i = 1, \dots, k-1$, where $k \geq 2$ and the following conditions hold:

(i) $\prod_{i=1}^{n} p_i \leq \prod_{i=1}^{n} p_i^*$;

(ii) $\begin{bmatrix} \alpha_1^* & \dots & \alpha_n^* \\ \lambda_1^* & \dots & \lambda_n^* \end{bmatrix} = \begin{bmatrix} \alpha_1 & \dots & \alpha_n \\ \lambda_1 & \dots & \lambda_n \end{bmatrix} T_1 \dots T_k$;

Then, for $(\boldsymbol{\alpha}, \boldsymbol{\lambda}) \in \mathcal{F}_n$, we have $Y_{1:n}^* \geq_{st} (\geq_{sl}) Y_{1:n}$.

Proof: By Lemma 12, assumption (ii) implies $X_{1:n}^* \geq_{st} X_{1:n}$. Now, in light of this observation and (1), the required result follows immediately from assumption (i). ■

The following corollary is a direct consequence of Theorem 4 which shows that larger stochastic order smallest claim amount leads to the desirable property of uniformly larger VaR.

Corollary 2: Suppose $X_{\lambda_1}, \dots, X_{\lambda_n}$ are independent non-negative random variables with $X_{\lambda_i} \sim GE(\alpha_i, \lambda_i)$, $i = 1, \dots, n$. Further, suppose $I_{p_1}, \dots, I_{p_n}$ are independent Bernoulli random variables, independent of the X_{λ_i}'s, with $E(I_{p_i}) = p_i$, $i = 1, \dots, n$. Assume that Assume that $(\boldsymbol{\alpha}, \boldsymbol{\lambda}) \in \mathcal{F}_n$, and the following conditions hold:

(i) $\prod_{i=1}^{n} p_i \leq \prod_{i=1}^{n} p_i^*$;

(ii) $\begin{bmatrix} \alpha_1 & \dots & \alpha_n \\ \lambda_1 & \dots & \lambda_n \end{bmatrix} = \begin{bmatrix} \alpha_1^* & \dots & \alpha_n^* \\ \lambda_1^* & \dots & \lambda_n^* \end{bmatrix} T$;

Then, we have $VaR[Y_{1:n}^*; p] \geq VaR[Y_{1:n}; p]$. ■

The following example provides an illustration of Theorem 6.

Example 2: Suppose $X_{\lambda_1}, X_{\lambda_2}, X_{\lambda_3}$ are independent non-negative random variables with $X_{\lambda_i} \sim GE(\alpha_i, \lambda_i)$, $i = 1, 2, 3$. Further, suppose $I_{p_1}, I_{p_2}, I_{p_3}$ are independent Bernoulli random variables, independent of the X_{λ_i} 's, with $E(I_{p_i}) = p_i$, $i = 1, 2, 3$. Also, suppose that $(\alpha_1, \alpha_2, \alpha_3) = (6, 10, 8)$, $(\lambda_1, \lambda_2, \lambda_3) = (7, 3, 4)$, $(\alpha_1^*, \alpha_2^*, \alpha_3^*) = (8.776, 6.64, 8.584)$, $(\lambda_1^*, \lambda_2^*, \lambda_3^*) = (3.652, 6.28, 4.068)$,

$(p_1, p_2, p_3) = (0.1, 0.2, 0.3), (p_1^*, p_2^*, p_3^*) = (0.1, 0.25, 0.4)$. Consider the T −transform matrices T_1, T_2 and T_3 as follows:

$$T_1 = 0.6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 0.4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$T_2 = 0.2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 0.8 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$T_3 = 0.1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 0.9 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

It is then easy to observe that $(\boldsymbol{\alpha}, \boldsymbol{\lambda})$, $(\boldsymbol{\alpha}, \boldsymbol{\lambda})T_1$ and $(\boldsymbol{\alpha}, \boldsymbol{\lambda})T_1T_2$ in $\mathcal{F}_3$ and $(\alpha^*, \lambda^*) = (\boldsymbol{\alpha}, \boldsymbol{\lambda})T_1T_2T_3$ which based on Theorem 6, yields $Y_{1:3}^* \geq_{st} (\geq_{sl}) Y_{1:3}$. ■

5. Conclusions and Suggestions

Suppose $X_{\lambda_1}, \dots, X_{\lambda_n}$ is a set of non-negative random variables with X_{λ_i} having the distribution function generalized exponential for $i = 1, \dots, n$, and $I_{p_1}, \dots, I_{p_n}$ are independent Bernoulli random variables, independent of the X_i's, with $E(I_{p_i}) = p_i, i = 1, \dots, n$. Let $Y_i = X_{\lambda_i} I_{p_i}$, for $i = 1, \dots, n$. It is of interest to note that in actuarial science, it corresponds to the claim amount in a portfolio of risks. In this paper, under some conditions and by using the concept of vector majorization and related orders, we have discussed stochastic comparison of the smallest claim amounts in the sense of the usual stochastic. The usual stochastic order between smallest claim amounts when the matrix of parameters $(\boldsymbol{\alpha}, \boldsymbol{\lambda})$ changes to another matrix in a mathematical sense is also discussed. The main question which arises here is how we can extend these results to the largest claim amounts. We are currently looking into this problem and hope to report the findings in a future paper.

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